Math 72 7.7 Complex Numbers

Objectives

1) Evaluate the basic complex number i
2) Use i to evaluate square roots of negative numbers
3) Add, Subtract complex numbers.
4) Mutiply complex numbers
5) Find the complex conjugate of a number.
6) Divide complex numbers.
7) Powers of i (optional)

We said before that $\sqrt{neg\#}$ = not a real number
Now we want to be more specific about the actual answer.

We will define $\sqrt{-1} = i$

- i is always lower case
- i is called the <u>imaginary unit</u>
- i is the most basic form of a number that is not real.

Simplify.

yes ① $\sqrt{-25}$ notice radicand is a negative number

$= \sqrt{25\cdot(-1)}$ separate negative 1 from the positive number

$= \sqrt{25}\cdot\sqrt{-1}$ use product property to separate

$= \boxed{5i}$ simplify the square root
replace $\sqrt{-1}$ by i

yes ② $\sqrt{-2}$

$= \sqrt{2\cdot(-1)}$

$= \sqrt{2}\cdot\sqrt{-1}$

$= \sqrt{2}\cdot i$

$= \boxed{i\sqrt{2}}$

We usually write i last – *except*
when a square root remains.
Then we write i before the radical

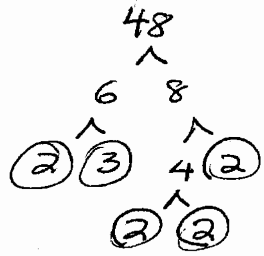
CAUTION
The i is outside the $\sqrt{\ }$
Do Not WRITE $\sqrt{i}$

yes ③ $\sqrt{-48}$

$= \sqrt{16\cdot3\cdot(-1)}$ separate -1 and perfect square factor

$= \sqrt{16}\cdot\sqrt{3}\cdot\sqrt{-1}$

$= 4\sqrt{3}\,i \quad = \boxed{4i\sqrt{3}}$



$48 = 2^4\cdot3$
$= 4^2\cdot3$
$= 16\cdot3$

When a number is i multiplied by any number (rational or irrational), but no other terms, we call this number a <u>purely imaginary number</u>.

ex: $5i$, $i\sqrt{2}$, $-6i$, $4i\sqrt{3}$ are all purely imaginary numbers.

We can add a real number to a purely imaginary number

ex. $2+5i$, $1+i\sqrt{2}$, $3-6i$, $-4+4i\sqrt{3}$

These numbers have a <u>real part</u> and an <u>imaginary part</u>.

Simplify

NO ④ $3-\sqrt{-16}$

$$= 3-\sqrt{16\cdot(-1)}$$

$$= 3-\sqrt{16}\cdot\sqrt{-1}$$

$$= \boxed{3-4i}$$

We did nothing with the real part 3 except re-copy it.

NO ⑤ $5+\sqrt{-12}$

$$= 5+\sqrt{4\cdot3\cdot(-1)}$$

$$= 5+\sqrt{4}\cdot\sqrt{3}\cdot\sqrt{-1}$$

$$= 5+2\sqrt{3}\,i$$

$$= \boxed{5+2i\sqrt{3}}$$

YES ⑥ $\dfrac{15-\sqrt{-75}}{5}$

$$= \frac{15-\sqrt{25\cdot(-1)\cdot3}}{5}$$

simplify $\sqrt{-75}$ as before

$$= \frac{15-\sqrt{25}\cdot\sqrt{-1}\cdot\sqrt{3}}{5}$$

con't ⇒

$$= \frac{15 - 5i\sqrt{3}}{5}$$

divide each term,
the real part 15 and
the imaginary part $5i\sqrt{3}$,
by 5

$$= \frac{15}{5} - \frac{5i\sqrt{3}}{5}$$

$$= \boxed{3 - i\sqrt{3}}$$

When we begin some problems, we may not know if the answer will be

- entirely a real number
- purely an imaginary number

or
- a real part and an imaginary part.

So we say that any number that can be written as

$a + bi$, which is called standard form,

where a is a real part and bi is an imaginary part is called a complex number,

EVEN IF $a = 0$
OR $b = 0$.

Write in the form $a+bi$:

yes ⑦ $2i = \boxed{0 + 2i}$ $a=0, b=2$

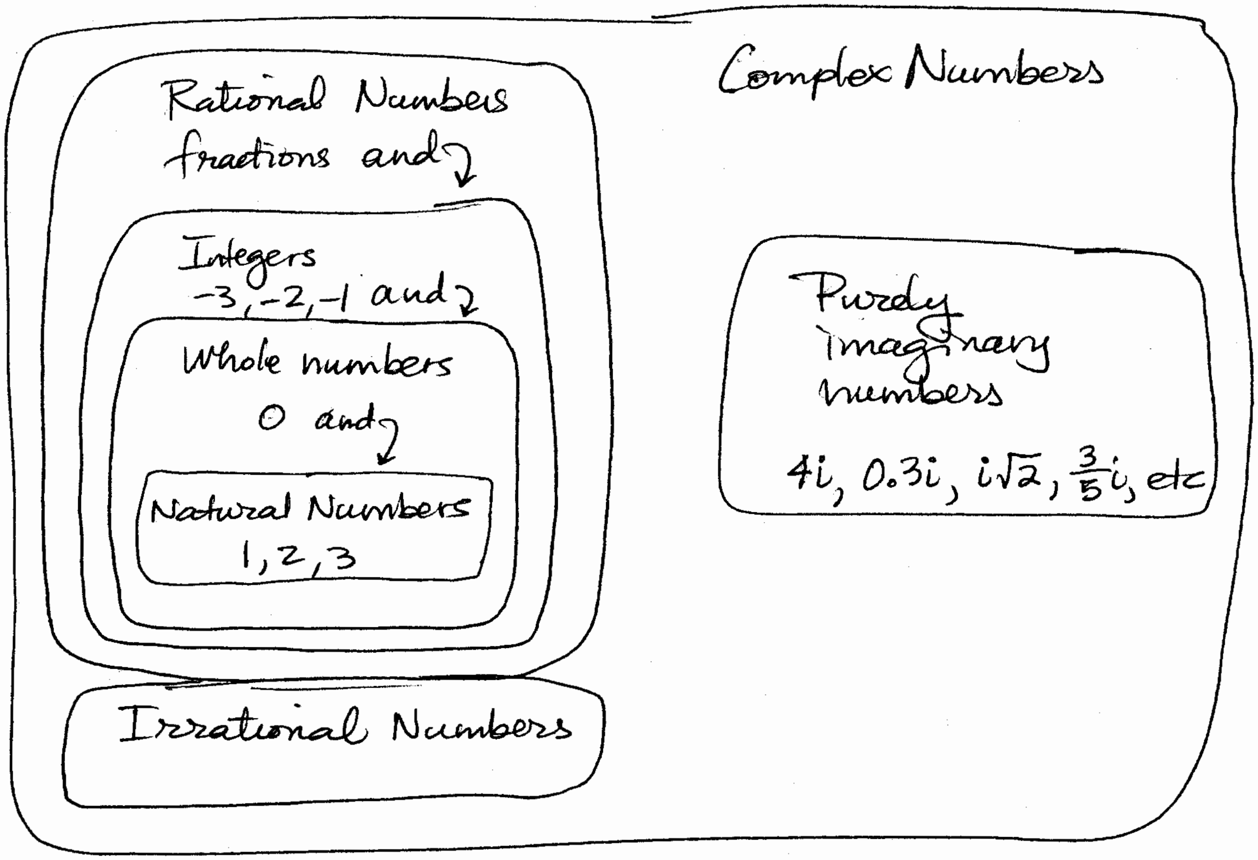
yes ⑧ $\sqrt{3} = \boxed{\sqrt{3} + 0i}$ $a=\sqrt{3}, b=0$

No ⑨ $6 = \boxed{6 + 0i}$ $a=6, b=0$

No ⑩ $\frac{1}{3} = \boxed{\frac{1}{3} + 0i}$ $a=\frac{1}{3}, b=0$

So the set of complex numbers includes all the real numbers and all the purely imaginary numbers as well as the numbers which have both real and imaginary parts.

This means that the set of complex numbers is the biggest set (has more numbers in it) we have discussed. In fact, every set we have discussed is contained in the set of complex numbers!



Recall:

$$\text{Rational} \underset{\substack{\Uparrow \\ \text{union, join together}}}{\cup} \text{Irrational} = \text{Real}$$

When we add or subtract complex numbers, we will add/subtract real parts to real parts and purely imaginary parts to purely imaginary parts.
{This is effectively the same as combining like terms}

Simplify

yes ⑪ $(2+3i)+(-6+7i)$

$= 2+3i+-6+7i$

$= 2-6+3i+7i$

$= \boxed{-4+10i}$

The parentheses are organizational only— they alert you that the first term is real and the second is imaginary.

Simplify.

yes ⑫ $(5+\sqrt{-36})+(2-\sqrt{-49})$

$= 5+\sqrt{36\cdot(-1)} + 2 - \sqrt{49\cdot(-1)}$ simplify radicals first

$= 5+6i+2-7i$ combine like parts

$= 5+2+6i-7i$

$= \boxed{7-i}$

yes ⑬ $(-1+5i)-(8+3i)$ just as we distribute the negative when subtracting polynomials, we do it here, too.

$= -1+5i-8-3i$

$= -1-8+5i-3i$

$= \boxed{-9+2i}$

yes ⑭ $(3+\sqrt{-16})-(-2-\sqrt{-100})$

$= (3+\sqrt{16\cdot(-1)})-(-2-\sqrt{100\cdot(-1)})$ simplify

$= (3+4i)-(-2-10i)$ dist neg

$= 3+4i+2+10i$ combine

$= \boxed{5+14i}$

If $i=\sqrt{-1}$
Then $i^2=(\sqrt{-1})^2=-1$

CAUTION: Never write a final answer using i^2. Simplify $i^2=-1$.

Yes ⑮ $2i(5-3i)$

This is a multiply.
We distribute $2i$ to both terms

$= 2i\cdot5 - 2i\cdot3i$

$= 2\cdot5\cdot i - 2\cdot3\cdot i\cdot i$ Use the commutative property of multiplication to change the order

$= 10i - 6\cdot i^2$

$= 10i - 6(-1)$

$= 10i+6 = \boxed{6+10i}$

Simplify

yes ⑯ $(5-2i)(-1+3i)$

This is a multiply.
Two terms × Two terms
means FOIL.

$$= \underset{F}{5(-1)} + \underset{O}{5\cdot 3i} - \underset{I}{2i(-1)} - \underset{L}{(2i)(3i)}$$

$$= -5 + 15i + 2i - 6i^2$$

$$= -5 + 17i - 6(-1)$$

$$= -5 + 6 + 17i$$

$$= \boxed{1+17i}$$

Note: $i=\sqrt{-1}$ is imaginary
but $i^2=(-1)$ is real.

yes ⑰ $\sqrt{-49}\cdot\sqrt{-4}$

$$= \sqrt{49\cdot(-1)} \cdot \sqrt{4(-1)}$$

$$= (7i)\cdot(4i)$$

$$= 28i^2$$

$$= 28(-1)$$

$$= \boxed{-28}$$

Very important:
$\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$ ONLY WORKS when $\sqrt{a}$ and $\sqrt{b}$ are real numbers
$\sqrt{a}\cdot\sqrt{b} \neq \sqrt{ab}$ when $\sqrt{a}$ and $\sqrt{b}$ are both imaginary.

THIS MEANS:
⇒ Always simplify $\sqrt{neg}$ 1st then multiply i by i

yes ⑱ $(3+\sqrt{-25})(1-\sqrt{-9})$

$$= (3+\sqrt{25(-1)})(1-\sqrt{9(-1)})$$

$$= (3+5i)(1-9i)$$

$$= 3 - 27i + 5i - 45i^2$$

$$= 3 - 22i - 45(-1)$$

$$= 3 - 22i + 45$$

$$= \boxed{48-22i}$$

Simplify.

No ⑲ $(3-4i)^2$

↑
subtract means FoIL

$= (3-4i)(3-4i)$

$= 3\cdot 3 - 3\cdot 4i - 4i\cdot 3 + 4i\cdot 4i$ FoIL

$= 9 - 12i - 12i + 16i^2$ mult

$= 9 - 24i + 16(-1)$ combine imaginary parts

subst $i^2 = -1$

$= 9 - 24i - 16$

$= \boxed{-7 - 24i}$ a+bi form

yes ⑳ $(-3-\sqrt{-9})^2$

↑ simplify i first

$\sqrt{-9} = \sqrt{9(-1)} = \sqrt{9}\cdot\sqrt{-1} = 3i$

$= (-3-3i)^2$

↑
subtract means FoIL

$= (-3-3i)(-3-3i)$

$= 9 + 9i + 9i + 9i^2$

$= 9 + 18i + 9(-1)$

$= 9 + 18i - 9$

$= \boxed{18i}$

Multiply

yes ㉑ $(4+3i)(4-3i)$

← Notice: These two complex numbers both have imaginary parts.

$= 16 - 12i + 12i - 9i^2$ FOIL

$= 16 - 9(-1)$

$= 16 + 10$

$= \boxed{26}$

← But when we complete the multiplication, the result is a real number because the middle terms cancel out.

Rationalize

yes ㉒ $\frac{3-2\sqrt{5}}{3\sqrt{2}}$ ← see next pages

yes ㉓ $\frac{3-2\sqrt{5}}{5+3\sqrt{2}}$ If time ㉔ $\frac{2+\sqrt{3}}{\sqrt{5}-\sqrt{2}}$

It's essentially a difference of squares

$$\left\{\begin{aligned}(x-y)(x+y) &= x^2 + xy - xy - y^2\\ &= x^2 - y^2\end{aligned}\right\}$$

Because this pattern results in a real result, these two complex numbers, $4+3i$ and $4-3i$, are related to each other in a special way.

They are called <u>complex conjugates</u>.

$4+3i$ is the complex conjugate of $4-3i$

$4-3i$ is the complex conjugate of $4+3i$.

> Notice: The real parts are exactly the same.
> the imaginary parts have opposite signs.

Rationalize the denominators.

yes ㉒ $\frac{3-2\sqrt{5}}{3\sqrt{2}}$

$\sqrt{2}$ is the irrational part of a monomial denominator

$$= \frac{(3-2\sqrt{5})}{(3\sqrt{2})} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{3\sqrt{2} - 2\sqrt{10}}{3 \cdot 2}$$ distribute $\sqrt{2} \cdot \sqrt{2} = 2$

$$= \boxed{\frac{3\sqrt{2} - 2\sqrt{10}}{6}}$$

or

$$= \frac{3\sqrt{2}}{6} - \frac{2\sqrt{10}}{6}$$

$$= \boxed{\frac{\sqrt{2}}{2} - \frac{\sqrt{10}}{3}}$$

yes ㉓ $\frac{3-2\sqrt{5}}{5+3\sqrt{2}}$

binomial denominator, must use conjugate

$$= \frac{(3-2\sqrt{5})}{(5+3\sqrt{2})} \cdot \frac{(5-3\sqrt{2})}{(5-3\sqrt{2})}$$ ← FOIL ← FOIL

$$= \frac{15 - 9\sqrt{2} - 10\sqrt{5} + 6\sqrt{10}}{25 \underbrace{- \text{stuff} + \text{stuff}}_{\text{adds to } 0} - 9 \cdot 2}$$

$$= \boxed{\frac{15 - 9\sqrt{2} - 10\sqrt{5} + 6\sqrt{10}}{7}}$$

no gain to split up $\frac{15}{7} - \frac{9\sqrt{2}}{7} - \frac{10\sqrt{5}}{7} + \frac{6\sqrt{10}}{7}$

Rationalize the denominator

If time ㉔ $\dfrac{2+\sqrt{3}}{\sqrt{5}-\sqrt{2}}$

conjugate of denominator

$$= \frac{(2+\sqrt{3})}{(\sqrt{5}-\sqrt{2})} \cdot \frac{(\sqrt{5}+\sqrt{2})}{(\sqrt{5}+\sqrt{2})}$$

$$= \frac{2\sqrt{5} + 2\sqrt{2} + \sqrt{15} + \sqrt{6}}{5 \underbrace{+ \text{stuff} - \text{stuff}}_{\text{adds to } 0} - 2}$$

$$= \boxed{\frac{2\sqrt{5} + 2\sqrt{2} + \sqrt{15} + \sqrt{6}}{3}}$$

no gain to split up $\dfrac{2\sqrt{5}}{3} + \dfrac{2\sqrt{2}}{3} + \dfrac{\sqrt{15}}{3} + \dfrac{\sqrt{6}}{3}$.

Find the complex conjugate of each number.

yes ㉕ $-3+5i$

complex conjugate is $\boxed{-3-5i}$

$\Uparrow$ same real part $\quad$ $\Uparrow$ opposite imaginary part

yes ㉖ $4i$

$4i = 0+4i$

complex conjugate is $0-4i = \boxed{-4i}$

No ㉗ 2

$2 = 2+0i$

complex conjugate is $2-0i = \boxed{2}$ $\leftarrow$ complex conjugate is pretty boring for purely real numbers.

An expression which has i in the denominator is not simplified --

Ex: $\dfrac{-3+i}{5+3i}$ actually means $\dfrac{-3+\sqrt{-1}}{5+\sqrt{-9}}$.

Though this denominator is complex, it can been seen as a square root.

Just as we rationalized denominators in 9.6, we want to remove this i from denominator also.

We will do this using the same techniques as 9.6, only with i.

* CAUTION * Though this process looks like rationalizing AND we do almost no dividing in the process, the instruction will say: "Divide".

Divide.

yes ㉘ $\dfrac{-3+i}{5+3i}$

Goal: Rewrite as an equivalent expression with a purely real number in denominator

Method: Multiply by $1 = \dfrac{\text{complex conjugate of denom}}{\text{complex conjugate of denom}}$

$$= \frac{-3+i}{5+3i} \cdot \frac{5-3i}{5-3i}$$

step 1: Find complex conjugate of denom.

Multiply by 1.

dd parentheses

$$= \frac{(-3+i)(5-3i)}{(5+3i)(5-3i)}$$

$$= \frac{-15+9i+5i-3i^2}{25-15i+15i-9i^2}$$

step 2: FOIL numerator

FOIL denominator

$$= \frac{-15+14i-3(-1)}{25-9(-1)}$$

step 3: Combine

Simplify $i^2=-1$.

$$= \frac{-15+3+14i}{25+9}$$

* check *

If you don't have a single real number in the denominator, try again.

$$= \frac{-12+14i}{34}$$

$$= \frac{-12}{34} + \frac{14}{34}i$$

$$= \boxed{\frac{-6}{17} + \frac{7}{17}i}$$

step 4: Divide each term by the denominator and reduce the resulting fractions.

This is a+bi form, a clearly separate real part $\left(\frac{-6}{17}\right)$ and imaginary part $\left(\frac{7}{17}i\right)$.

Divide

yes ㉙ $\frac{4+6i}{-9i}$ ← monomial, mult by i only

$$= \frac{(4+6i)}{-9i} \cdot \frac{i}{i}$$

$$= \frac{4i+6i^2}{-9i^2}$$

$$= \frac{4i+6(-1)}{-9(-1)}$$

$$= \frac{-6+4i}{9}$$

$$= \frac{-6}{9} + \frac{4}{9}i$$

$$= \boxed{\frac{-2}{3} + \frac{4}{9}i}$$

Option 2:

$$\frac{4}{-9i} + \frac{6i}{-9i}$$

$$\frac{4 \cdot i}{-9i \cdot i} - \frac{6}{9}$$

$$\frac{4i}{-9(i)^2} - \frac{2}{3}$$

$$\frac{4i}{9} - \frac{2}{3}$$

$$\boxed{-\frac{2}{3} + \frac{4}{9}i}$$

a+bi form!

㉚ $\frac{2-3i}{7-8i}$

$$= \frac{(2-3i)}{(7-8i)} \cdot \frac{(7+8i)}{(7+8i)}$$ FOIL using conjugate of denominator / FOIL

$$= \frac{14+16i-21i-24i^2}{49 + \underbrace{56i-56i}_{\text{adds to } 0} - 64i^2}$$

$$= \frac{14-5i-24(-1)}{49-64(-1)}$$

$$= \frac{14-5i+24}{49+64}$$

$$= \frac{38-5i}{113} = \boxed{\frac{38}{113} - \frac{5i}{113}}$$

Divide.

③1 $\dfrac{3+4i}{2i}$

We have options because the denom has only one term.

Option 1: Do it the same way, mult by 1

Less desirable

$\dfrac{3+4i}{2i} \cdot \dfrac{-2i}{-2i}$

complex conjugate has same real part but opposite imaginary part

$= \dfrac{-2i(3+4i)}{-4i^2}$

distribute

$= \dfrac{-6i-8i^2}{-4(-1)}$

simplify $i^2=-1$

$= \dfrac{-6i-8(-1)}{4}$

divide each term

$= \dfrac{-6}{4}i + \dfrac{8}{4}$

reduce

$= -\dfrac{3}{2}i + 2$

write a+bi form

$= \boxed{2-\dfrac{3}{2}i}$

Option 2: Multiply by $1 = \dfrac{i}{i}$ only

$\dfrac{(3+4i)}{2i} \cdot \dfrac{i}{i}$

$= \dfrac{3i+4i^2}{2i^2}$

$= \dfrac{3i+4(-1)}{2(-1)}$

cont ⇒

$$= \frac{3i-4}{-2}$$

Remember: A negative in denominator is not simplified.

$$= \frac{3}{-2}i - \frac{4}{-2}$$

$$= \frac{-3}{2}i + 2$$

$$= \boxed{2-\frac{3}{2}i}$$

Option 3: Divide each term by $2i$.

$$\frac{3+4i}{2i}$$

$$= \frac{3}{2i} + \frac{4i}{2i}$$

$$= \frac{3}{2i} \cdot \frac{i}{i} + 2$$

still have to multiply the 1st term by $\frac{i}{i}$

$$= \frac{3i}{2i^2} + 2$$

$$= \frac{3i}{2(-1)} + 2$$

$$= \frac{3i}{-2} + 2$$

$$= \boxed{2-\frac{3}{2}i}$$

(32) Evaluate powers of i

a) $i^2 = \boxed{-1}$

b) $i^4 = (i^2)(i^2) = (-1)(-1) = \boxed{1}$

c) $i^3 = (i^2)(i) = -1(i) = \boxed{-i}$

d) $i^5 = (i\ )(i) = 1 \cdot i = \boxed{i}$

e) $i^6 = i^4 \cdot i^2 = 1 \cdot (-1) = \boxed{-1}$

f) $i^7 = i^4 \cdot i^3 = 1(-i) = \boxed{-i}$

g) $i^8 = i^4 \cdot i^4 = 1 \cdot 1 = \boxed{1}$

* MEMORIZE

$i^1 = i$

$i^2 = -1$

$i^3 = -i$

$i^4 = 1$

See a pattern?

$$\begin{array}{cccc} \begin{bmatrix} i \\ i^5 \\ i^9 \\ i^{13} \end{bmatrix} & \begin{bmatrix} i^2 \\ i^6 \\ i^{10} \\ i^{14} \end{bmatrix} & \begin{bmatrix} i^3 \\ i^7 \\ i^{11} \\ i^{15} \end{bmatrix} & \begin{bmatrix} i^4 \\ i^8 \\ i^{12} \\ i^{16} \end{bmatrix} \\ \Uparrow & \Uparrow & \Uparrow & \Uparrow \\ \text{all} & \text{all} & \text{all} & \text{all} \\ i & -1 & -i & +1 \end{array}$$

Evaluate

㉝ i^{27}

$= i^{24} \cdot i^3$

$= (1)(-i)$

$= \boxed{-i}$

step 1: find nearest power of 4 that is less than exponent and rewrite using exponent laws

$a^{n+m} = a^n \cdot a^m$

step 2: Recall that $i^n = 1$ when n is a multiple of 4.

step 3: simplify other part from memory.

Add more

⑪ i^{19}

⑫ i^{102}

⑬ i^{53}

⑭ i^{-17}

⑮ i^{-27}

㉞ i^{38}

$= i^{36} \cdot i^2$

$= (1)(-1)$

$= \boxed{-1}$

○ $i^{40} = \boxed{1}$

㉟ i^{19}

$$\begin{array}{r} 4 \\ 4\overline{)19} \\ \underline{16} \\ 3 \end{array}$$

$= (i^4)^4 \cdot i^3$

$= 1 \cdot i^3$

$= \boxed{-i}$

㊱ i^{102}

$$\begin{array}{r} 25 \\ 4\overline{)102} \\ \underline{8} \\ 22 \\ \underline{20} \\ 2 \end{array}$$

$= (i^4)^{25} \cdot i^2$

$= 1 \cdot (-1)$

$= \boxed{-1}$

㊲ i^{53}

$$\begin{array}{r} 13 \\ 4\overline{)53} \\ \underline{4} \\ 13 \\ \underline{12} \\ 1 \end{array}$$

$= (i^4)^{13} \cdot i$

$= 1i$

$= \boxed{i}$

㊳ i^{-17}

$$\begin{array}{r} 4 \\ 4\overline{)17} \\ \underline{16} \\ 1 \end{array}$$

$= \frac{1}{i^{16} \cdot i}$

$= \frac{1}{1 \cdot i} \cdot \frac{i}{i}$

$= \frac{i}{i^2}$

$= \frac{i}{-1}$

$= \boxed{-i}$

㊴ i^{-27}

$$\begin{array}{r} 6 \\ 4\overline{)27} \\ \underline{24} \\ 3 \end{array}$$

$= \frac{1}{(i^4)^6 i^3}$

$= \frac{1}{1 \cdot (-i)}$

$= \frac{1}{-i} \cdot \frac{i}{i}$

$= \frac{i}{-(-1)}$

$= \boxed{i}$